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ON CONTINUED FRACTIONS IN NON-COMMUTATIVE QUANTITIES.

By J. H. M. Wedderburn.

The object of this note is to investigate the properties of simple continued fractions when the terms are not necessarily commutative with one another. The terms, for instance, may be the product of a function of x and differential operator D_x; or they may be matrices; or in fact any functional operator or hypercomplex quantity for which addition obeys the ordinary laws of algebra and multiplication is associative and distributive. In the demonstrations it is always assumed that an inverse exists, but, as the relations given are all integral identities, the truth of the theorems does not depend on the existence of an inverse: the use of the inverse could therefore no doubt have been avoided, as by studying Euclid's algorithm in place of continued fractions for instance, but it does not seem that there would be any material gain in doing so.

1. Consider the continued fraction

$$a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \cdots.$$

When we come to form the convergents of this expression, it is immediately obvious that there are two principal ways of writing them according as we put the denominator of the convergent on the right or on the left of numerator. For instance,

$$a_1 + \frac{1}{a_2} = (a_1a_2 + 1)a_2^{-1} = a_2^{-1}(a_2a_1 + 1),$$

$$a_1 + \frac{1}{a_2} + \frac{1}{a_3} = (a_1a_2a_3 + a_1 + a_3)(a_2a_3 + 1)^{-1},$$

$$= (a_3a_2 + 1)^{-1}(a_3a_2a_1 + a_1 + a_3),$$

and so on.

When the denominator is on the right, the nth convergent will be denoted by $p_nq_n^{-1}$, and when the denominator is on the left by $\bar{q}_n^{-1}\bar{p}_n$.

Theorem 1.

$$p_n = p_{n-1}a_n + p_{n-2}, \qquad q_n = q_{n-1}a_n + q_{n-2}$$

$$\bar{p}_n = a_n\bar{p}_{n-1} + \bar{p}_{n-2}, \qquad \bar{q}_n = a_n\bar{q}_{n-1} + \bar{q}_{n-2}$$

The proof is exactly the same as in ordinary algebra.

Assume for a particular value of n that

$$p_n q_n^{-1} = (p_{n-1}a_n + p_{n-2})(q_{n-1}a_n + q_{n-2})^{-1},$$

then

$$\begin{aligned} p_{n+1}q_{n+1}^{-1} &= [p_{n-1}(a_n + a_{n+1}^{-1}) + p_{n-2}][q_{n-1}(a_n + a_{n+1}^{-1}) + q_{n-2}]^{-1} \\ &= [(p_{n-1}a_n + p_{n-2})a_{n+1} + p_{n-1}]a_{n+1}^{-1}[\{(q_{n-1}a_n + q_{n-2})a_{n+1} \\ &\qquad + q_{n-1}\}a_{n+1}^{-1}]^{-1} \\ &= (p_n a_{n+1} + p_{n-1})(q_{n+1}a_{n+1} + q_{n-1})^{-1}. \end{aligned}$$

But the theorem is readily verified for $n = 1, 2$ and hence it is true for all values. The second part of the theorem follows in exactly the same way.

THEOREM 2. $\quad \bar{q}_n p_n - \bar{p}_n q_n = 0.$

For

$$p_n q_n^{-1} = \bar{q}_n^{-1} p_n$$

since each is equal to the value of the nth convergent.

THEOREM 3.

$$\begin{aligned} &(i)\ \bar{q}_{n-1}p_n - \bar{p}_{n-1}q_n = (-1)^n, \\ &(ii)\ \bar{q}_n p_{n-1} - \bar{p}_n q_{n-1} = (-1)^{n+1}, \\ &(iii)\ p_n\bar{q}_{n-1} - p_{n-1}\bar{q}_n = (-1)^n, \\ &(iv)\ q_n\bar{p}_{n-1} - q_{n-1}\bar{p}_n = (-1)^{n+1}. \end{aligned}$$

Let

$$\Delta_n = \bar{q}_{n-1}p_n - \bar{p}_{n-1}q_n, \qquad \bar{\Delta}_n = \bar{q}_n p_{n-1} - \bar{p}_n q_{n-1}.$$

By Theorem 1

$$\begin{aligned} \Delta_n &= \bar{q}_{n-1}(p_{n-1}a_n + p_{n-2}) - \bar{p}_{n-1}(q_{n-1}a_n + q_{n-2}) \\ &= (\bar{q}_{n-1}p_{n-1} - \bar{p}_{n-1}q_{n-1})a_n + \bar{q}_{n-1}p_{n-2} - \bar{p}_{n-1}q_{n-2} \\ &= \bar{q}_{n-1}p_{n-2} - \bar{p}_{n-1}q_{n-2} = \bar{\Delta}_{n-1}, \end{aligned}$$

since, by Theorem 2,

$$\bar{q}_{n-1}p_{n-1} - \bar{p}_{n-1}q_{n-1} = 0.$$

Similarly $\bar{\Delta}_n = \Delta_{n-1}$. Hence Δ_n equals Δ_2 if n is even, and $\bar{\Delta}_2$ if n is odd. But

$$\Delta_2 = a_1a_2 + 1 - a_1a_2 = 1, \qquad \bar{\Delta}_2 = a_2a_1 - (a_2a_1 + 1) = -1.$$

Hence $\Delta_n = (-1)^n$: similarly $\bar{\Delta}_n = (-1)^{n+1}$.

Again if

$$\Delta_n' = p_n\bar{q}_{n-1} - p_{n-1}\bar{q}_n,$$

then

$$\begin{aligned} \Delta_n' &= (p_{n-1}a_n + p_{n-2})\bar{q}_{n-1} - p_{n-1}(a_n\bar{q}_{n-1} + \bar{q}_{n-2}) \\ &= p_{n-2}\bar{q}_{n-1} - p_{n-1}\bar{q}_{n-2} = -\Delta_{n-1}' \end{aligned}$$

and $\Delta_2' = 1$: hence $\Delta_n' = (-1)^n$.

(iv) is proved in exactly the same fashion.

Since p_n, q_n, $\bar{p}_n$, $\bar{q}_n$ are integral when $a_1, a_2, \cdots$ are, it follows from these relations that p_n and q_n or $\bar{p}_n$ and $\bar{q}_n$ have in general no common factor, although owing to the indefinite nature of the a's no definite conclusion can be drawn. If for example they are differential operators mentioned in the introduction, then the result is valid. For if p_n and q_n have a right-handed factor in common, say

$$p_n = p_n'r, \qquad q_n = q_n'r,$$

then, if y is the dependent variable, the equations $p_n y = 0$ and $q_n y = 0$ have in common the solutions of $ry = 0$: but if y_1 were such a solution, then

$$(\bar{q}_{n-1}p_n - \bar{p}_{n-1}q_n)y_1 = 0,$$

which, by the foregoing theorem, is only possible if $y_1 = 0$.

THEOREM 4. $\quad p_n\bar{p}_{n-1} = p_{n-1}\bar{p}_n, \quad q_n\bar{q}_{n-1} = q_{n-1}\bar{q}_n.$

For

$$\begin{aligned}(p_n\bar{p}_{n-1})^{-1} = \bar{p}_{n-1}^{-1}p_n^{-1} &= (-1)^n\bar{p}_{n-1}^{-1}(\bar{q}_{n-1}p_n - \bar{p}_{n-1}q_n)p_n^{-1}\\ &= (-1)^n[\bar{p}_{n-1}^{-1}\bar{q}_{n-1} - q_np_n^{-1}]\\ &= (-1)^n[q_{n-1}p_{n-1}^{-1} - \bar{p}_n^{-1}\bar{q}_n]\\ &= (-1)^n\bar{p}_n^{-1}[\bar{p}_nq_{n-1} - \bar{q}_np_{n-1}]p_{n-1}^{-1}\\ &= \bar{p}_n^{-1}p_{-1}^{n-1} = (p_{n-1}\bar{p}_n)^{-1}.\end{aligned}$$

Similarly

$$q_n\bar{q}_{n-1} = q_{n-1}\bar{q}_n.$$

Finally, if C_n is the nth convergent, then as in ordinary algebra we have

THEOREM 5. $\quad C_n - C_{n-2} = (-1)^n\bar{q}_{n-2}^{-1}a_nq_n^{-1} = (-1)^n\bar{q}_n^{-1}a_nq_{n-2}^{-1}.$

For

$$\begin{aligned}C_n - C_{n-2} &= \bar{q}_{n-2}^{-1}(\bar{q}_{n-2}p_n - \bar{p}_{n-2}q_n)q_n^{-1}\\ &= (\bar{q}_{n-2}^{-1}[\bar{q}_{n-2}(p_{n-1}a_n + p_{n-2}) - \bar{p}_{n-2}(q_{n-1}a_n + q_{n-2})]q_n^{-1}\\ &= \bar{q}_{n-2}^{-1}[(\bar{q}_{n-2}p_{n-1} - \bar{p}_{n-2}q_{n-1})a_n + \bar{q}_{n-2}p_{n-2} - \bar{p}_{n-2}q_{n-2}]q_n^{-1}\\ &= (-1)^n\bar{q}_{n-2}^{-1}a_nq_n^{-1},\end{aligned}$$

and similarly

$$C_n - C_{n-2} = (-1)^n\bar{q}_n^{-1}a_nq_{n-2}^{-1}$$

also, from Theorem 3(i)

$$C_n - C_{n-1} = p_nq_n^{-1} - \bar{q}_{n-1}^{-1}p_{n-1} = (-1)^n\bar{q}_{n-1}^{-1}q_n^{-1}.$$

2. So far no special restriction has been imposed on the distributive

operators $a_1, a_2, \cdots$. We shall now assume that they are linear differential operators of the form $\Sigma r_n D^n$ where $D = d/dx$ and r_n are functions of x which possess derivatives so far as they are required in the processes employed below.

If p and q are two such operators, the degree of q being not higher than that of p, then we can by the process of ordinary division find operators $a, b, \bar{a}, \bar{b}$ such that

$$p = qa + b, \qquad p = \bar{a}q + \bar{b}.$$

The formal proof* of this is left to the reader. If therefore p and q have no factor in common on the right, we can express pq^{-1} uniquely as a simple terminating continued fraction of the kind we have considered above. If they have a common factor, this process of continued division determines the H.C.F. just as in the theory of ordinary polynomials.

Consider now the following system of ordinary linear differential equations in two dependent variables,

$$U_1 = py_1 + ry_2 = u_1, \qquad U_2 = qy_1 + sy_2 = u_2, \tag{1}$$

p, q, r and s being linear differential operators.

Let the H.C.F. of p and q be c and let $p = p_n c$, $q = q_n c$ where

$$pq^{-1} = p_n q_n^{-1} = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_n}.$$

Multiplying the first equation by $\bar{q}_n$ and the second by $\bar{p}_n$ and subtracting, and then performing the same operation with $\bar{q}_{n-1}$ and $\bar{p}_{n-1}$ in place of $\bar{q}_n$ and $\bar{p}_n$ and using the result obtained in Theorems 2 and 3, viz.,

$$\bar{q}_n p_n = \bar{p}_n q_n, \qquad \bar{q}_{n-1} p_n - \bar{p}_{n-1} q_n = (-1)^n,$$

we obtain a new system

$$\begin{aligned} V_1 &\equiv \bar{q}_n U_1 - \bar{p}_n U_2 = (\bar{q}_n r - \bar{p}_n s) y_2 = \bar{q}_n u_1 - \bar{p}_n u \equiv v_1, \\ V_2 &\equiv \bar{q}_{n-1} U_1 - \bar{p}_{n-1} U_2 = (-1)^n c y_1 + (\bar{q}_{n-1} r - \bar{p}_{n-1} s) y_2 = \bar{q}_{n-1} u_1 - \bar{p}_{n-1} u_2 \equiv v_2 \end{aligned} \tag{2}$$

and every solution of the original equation is a solution of (2) also.

Conversely

$$\begin{aligned} -p_{n-1} V_1 + p_n V_2 &= (p_n \bar{q}_{n-1} - p_{n-1} \bar{q}_n) U_1 + (p_{n-1} \bar{p}_n - p_n \bar{p}_{n-1}) U_2 = (-1)^{n+1} U_1, \\ q_{n-1} V_1 - q_n V_2 &= (q_{n-1} \bar{q}_n - q_n \bar{q}_{n-1}) U_1 + (q_n \bar{p}_{n-1} - q_{n-1} \bar{p}_n) U_2 = (-1)^{n+1} U_2, \end{aligned}$$

and exactly the same relations hold between v_1, v_2 and u_1, u_2. Hence every solution of the second system is also a solution of the first, i. e., the two are equivalent.

* See Pincherle, Le operazioni distributive, Bologna, 1901, p. 263.

Now (2) has the form

$$ay_2 = v_1,$$
$$(-1)^n cy_1 + by_2 = v_2,$$

hence we have reduced the solution of the original system to the solving of a chain of equations in one variable. The same method can evidently be used to reduce a system in any number of dependent variables to a sequence of equations in one variable just as in the theory of systems with constant coefficients.*

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* Cf. Chrystal, Trans. R. S. E., vol. 38 (1895), p. 163.